Geomechanical Analysis of Unbound Pavements Based on Shakedown Theory

By I. F. Collins,[1] Member, ASCE, and M. Boulbibane[2]

Abstract: A procedure for analyzing the mechanical response of an unbound pavement to the repeated loading of traffic is presented. The pavement is modeled as a layered elastic/plastic structure, and its response is described by the concepts of shakedown theory. A critical shakedown load is identified as the key design parameter. Pavements operating at higher loads will eventually fail, and those operating at loads less than critical may initially exhibit some distress but will eventually shakedown to a steady state. Estimates of this critical load, for different types of pavement, are found by studying various types of failure mechanisms, such as rut formation and subsurface slip. Optimization procedures are then used to determine the most likely form of failure for a particular pavement. The effects of self-weight, dual loads, moisture content, relative strengths of the various layers, and nonassociated plastic flow are studied. Some preliminary implications for pavement design are discussed.

INTRODUCTION

Among the various types of structures encountered by civil engineers, pavements are among the most poorly understood. This is, in part, due to the inherent complexities of the behavior of soils and granular aggregates encountered in most geotechnical problems. In the case of pavements, however, these material description problems are further exacerbated by the fact that the structure is subjected to repeated loads, and failure occurs by the gradual deterioration of the pavement, not by sudden collapse. Although design procedures are becoming more "mechanistic" and sophisticated in the use of computer programs to evaluate the design variables, little progress has been made in improving the basic mechanical model of the failure of the pavement. Specifically nearly all pavement design models concentrate on the elastic (i.e., resilient) behavior of the various pavement layers, including such "secondary" effects as stress dependent moduli and anisotropy. The elastic response by its very definition however cannot cause failure. In an elastic deformation the structure necessarily returns to its original state after the wheel load has passed. Nevertheless, most design models are based on the assumption that "failure" occurs when a certain elastic strain (e.g., the vertical compressive strain at the top of the subgrade) reaches a critical value. It is the irrecoverable (plastic, damage, or viscous) strains that, though very small at each load application, build up and eventually cause failure in the form of subsurface slip, rut formation, or surface cracking. There is hence a clear need to develop theoretical models, which include the irreversible response and attempt to model the actual failure mechanisms observed in pavements. A comprehensive review of the current understanding and practice of pavement engineering has been given by Brown (1996) in the 33rd Rankine Lecture to the British Geotechnical Society. He highlights the complexity of the problem, emphasizes that practice is lagging behind knowledge of the behavior of roading materials obtained from laboratory experiments and that theoretical models need to be improved.

A number of research papers have suggested the use of bearing capacity theory to model the plastic response of a pavement [see e.g., the recent paper by Oloo et al. (1997), and the references cited therein, and the review by Houlsby and Burd (1999)]. The soil strength parameters—the cohesion c and angle of internal friction ϕ—of the various pavement layers, replace the elastic moduli as the basic design variables. The model is rigid plastic, and the critical failure load can be calculated using limit analysis techniques [e.g., Michalowski and Shi (1995) and Purushothamaraj et al. (1974)], limit equilibrium methods as in Oloo et al. (1997), or finite elements as in Burd and Frydman (1997). However, this model presumes monotonic loading of the pavement and must be combined with some empirical damage rule, such as the well-known fourth power law, to predict the life of the pavement. The procedure advocated in the present paper allows the cyclic nature of the pavement loading to be modeled and incorporates both the elastic and plastic responses of the pavement to these loads.

In many other branches of engineering much progress has been made in understanding the response of structures to repeated loading and the nature of long-term failure and wear. The relatively new subject of "damage mechanics" is now providing many insights. Particularly relevant to the analysis of pavements is the progress that has been made in the understanding of wear processes of metal surfaces subjected to rolling and sliding loads [see, e.g., Johnson (1985, 1992), Ponter et al. (1985), Kapoor and Williams (1994a,b)]. Of particular relevance to pavement analysis is the studies on the wear of layered surfaces by Anderson and Collins (1995), Wong and Kapoor (1996), and Wong et al. (1997). These analyses all use the concept of shakedown theory to determine the long-term behavior of the surface. The basic assumption is that the structure can be modeled by an inhomogeneous elastic/plastic material, in which case the structure will eventually either shakedown [i.e., the ultimate response will be purely elastic (reversible)] or will fail in the sense that the structural response is always plastic (irreversible) however many times the load is applied. The critical load level separating these two types of behavior is termed the "shakedown load."

The possibility of using this shakedown load as the design load for pavements seems first to have been recognized by Booker and Sharp (1984) [see also Sharp (1985)]. Brett (1987) used a time series analysis to study the variation of roughness and serviceability indices of a number of sections of roads in New South Wales and concluded that after 8 years at least 40% were operating in a shaken-down state—so-called survivor pavements. Both Sharp and Brett quote a number of other field observations supporting the view that many pavements do shake down rather than deteriorate continuously. In

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Note. Discussion open until June 1, 2000. To extend the closing date one month, a written request must be filed with the ASCE Manager of Journals. The manuscript for this paper was submitted for review and possible publication on April 13, 1999. This paper is part of the ***Journal of Geotechnical and Geoenvironmental Engineering***, Vol. 126, No. 1, January, 2000. ©ASCE, ISSN 1090-0241/00/0001-0050–0059/$8.00 + $.50 per page. Paper No. 20703.

shakedown theory the structure is assumed to behave like a rate-independent elastic/plastic material; this concept is particularly appropriate for unbound or sealed gravel pavements, where the top viscous bitumen layer is very thin, serving only as a weatherproofing layer, and plays no structural role in the pavement response. This theory could also be claimed as being relevant to concrete pavements but is much less appropriate for asphaltic pavements where viscous and creep effects dominate and viscoelastic or viscoplastic models are more appropriate. The recent papers by Collop et al. (1995) and Ramsamooj et al. (1998) use such viscous models to analyze the failure of asphalt pavements.

The basic analysis of Booker and Sharp (1984) has been followed up by a number of investigators. Interest has centered around the calculation of the critical shakedown load. This is difficult to compute exactly, but lower and upper bounds can be obtained by methods similar to those employed in limit analysis for monotonic loadings. Lower-bound analysis based on consideration of stress fields has been given by Booker and Sharp (1984), Raad et al. (1988, 1989a,b), Hossain and Yu (1998), Yu and Hossain (1998), and Boulbibane and Weichert (1997). Such analyses use finite-element programs to calculate the elastic stress field and a linear programming procedure for finding the best lower bound for the shakedown load. Such procedures lead to very large-scale computations and to date all such calculations have assumed a 2D idealization, in which the quasi-circular loading area of a wheel is replaced by an infinite-strip loading under a infinitely long cylinder.

Collins and Cliffe (1987) pointed out that if instead one employed the dual kinematic upper-bound approach, which requires the optimization of load estimates made from a number of competing failure mechanisms, one could relatively easily solve the much more realistic 3D problem where the wheel load is applied over a circular patch. This approach also has the advantage that it introduces the pavement failure mode explicitly. These authors also showed that in the 2D problem their upper bounds coincided with the lower bounds given earlier by Booker and Sharp (1984), which were therefore the exact values. Detailed calculations of failures by subsurface slip in the direction of travel in layered pavements, subject to single or dual circular patch loadings, have been presented by Collins et al. (1993a,b), and some preliminary results for failures by rut formation, in which the pavement material in the top layer(s) is displaced sideways, perpendicular to the line of travel may be found in Collins and Boulbibane (1997, 1998a), with some design implication discussed in Collins and Boulbibane (1998b) and Boulbibane and Collins (1999).

The employment of shakedown theory enables us to determine the long-term behavior of the pavement without resorting to the tedious and computationally expensive process of computing the pavement's response to successive individual load applications.

The purpose of the present paper is to review the progress made to date in applying shakedown theory to pavement analysis and design, discuss its shortcomings, and indicate areas for future research.

WHAT IS SHAKEDOWN?

The various possible responses of an elastic-plastic structure to a cyclical load history are indicated schematically in Fig. 1. If the load level is sufficiently small, the response is purely elastic, no permanent strains are induced, and the structure returns to its original configuration after each load application. However, if the load level exceeds the elastic limit load, permanent plastic strains occur and the response of the structure to a second and subsequent loading cycle is different from the first. There are three basic causes of this effect:

1. Residual stresses are induced in the structure by the application of a load cycle, so that the total stress field induced in the second cycle is the sum of this residual field and that produced by the applied load.
2. Changing material properties (e.g., strain hardening or softening).
3. Changes to the geometry of the surface, as a consequence of the permanent strains induced there, can mean that the loading distribution may be different on the second application. In a pavement this would be true if a rut is forming, for example.

In the case of a pavement, effects 1 and 2 are deemed to be very important, but effect 3 is secondary and will be ignored in the analysis.

When the load exceeds the elastic limit the structure can exhibit three long-term responses depending on the load level. These are illustrated schematically in Fig. 1. After a finite number of load applications, the buildup of residual stresses and changing material properties can be such that the structures response is purely elastic, so that no further permanent strains occur. When this happens the structure is said to have "shakedown." In a pavement this could mean that some rutting, subsurface deformation, or cracking occurs but that after a certain time this deterioration ceases and no further structural

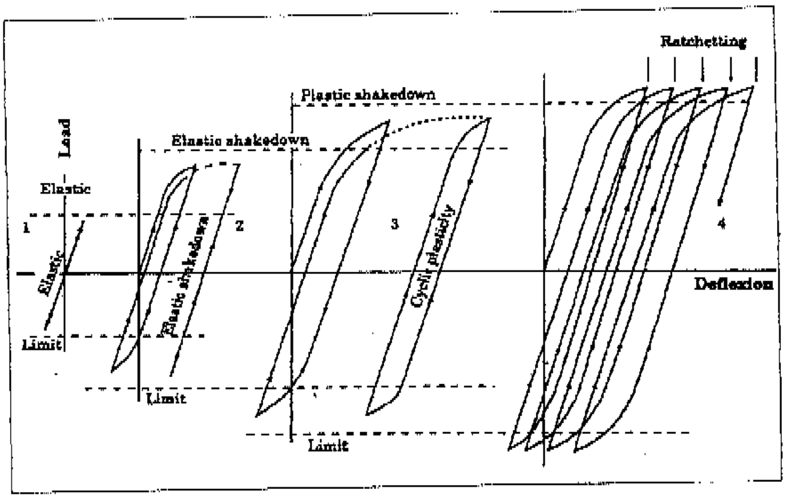

FIG. 1. Four Types of Response of Elastic/Plastic Structure to Repeated Loading Cycles

damage occurs. At still higher load levels however, shakedown does not occur, and either the permanent strains settle into a closed cycle—a situation known as "cyclic" or "alternating plasticity"—or they go on increasing indefinitely—known as "ratchetting." If either of these latter situations occurs, the structure will "fail."

The critical load level below which the structure shakes down and above which it fails is called the shakedown load and it is this parameter that is the key design load in the proposed procedure.

CALCULATION OF SHAKEDOWN LOAD

The direct calculation of the shakedown load is difficult. Instead lower and upper bounds are usually calculated using Melan's static or Koiter's kinematic theorems, respectively (Lubliner 1990). These procedures are similar to the familiar limit analysis techniques for failure under monotonic loading, except that now the elastic stress field needs to be known and included in the calculation. The material is assumed to be perfectly plastic with associated flow behavior.

Calculation of Lower Bounds

The lower-bound theorem states that a sufficient condition for shakedown to occur is that a time-independent, self-equilibrated, residual stress field can be found that, when added to the elastic stress field, produces a combined stress field that nowhere and at no-time violates the yield condition. In the pavement context a loading cycle consists of the loading patch moving from $x = -\infty$ to $x = +\infty$ (Fig. 2). Because all points (x, y, z), $-\infty < x < \infty$ undergo the same loading history, the residual stress distribution must be independent of the x-coordinate. The x-coordinate is essentially the time variable for this problem. The total load on the loading patch will be denoted by λP, where λ is a scalar load parameter. The induced linearly elastic stress field will hence be proportional to λ and will be denoted by $\lambda\sigma^e_{ij}$. Shakedown will hence occur if a residual stress field ρ_{ij} (which is independent of the x-coordinate, is in equilibrium, and has zero surface tractions) can be found such that the following inequality is satisfied everywhere in the pavement:

$$f(\lambda\sigma^e_{ij} + \rho_{ij}) \le 0 \qquad (1)$$

where f is the yield function. The maximum possible such value of λ will give the critical shakedown load.

Sharp and Booker (1984) calculated this maximum value of λ for single layer pavements under infinitely long strip loadings [Fig. 2(b)], where the wheel is replaced by an infinitely long cylinder. The resulting deformation is hence plane strain in the (x, z)-plane and there is only one no-zero residual stress component ρ_{xx} that is a function only of z. The optimization problem hence reduces to finding the critical depth at which equality in (1) provides the maximum value of λ. Sharp and Booker solved this problem by using an elegant "method of conics." This procedure however does not generalize easily to the 3D problem in which the loading is applied over one or more circular patches. There are now six residual stress components and the optimization problem is extremely large.

Calculation of Upper Bounds

Upper bounds to the shakedown load can be obtained from any virtual velocity field v^*_i, with associated strain rate field e^*_{ij} and plastic stress field σ^*_{ij}. We now present a proof of this result appropriate for pavement applications. Suppose σ^R_{ij} is the residual stress field induced in the pavement during its construction and before it is open to traffic. If the pavement shakes down once traffic loads are applied there must exist a self-equilibrated residual stress field ρ_{ij}, such that the total stress

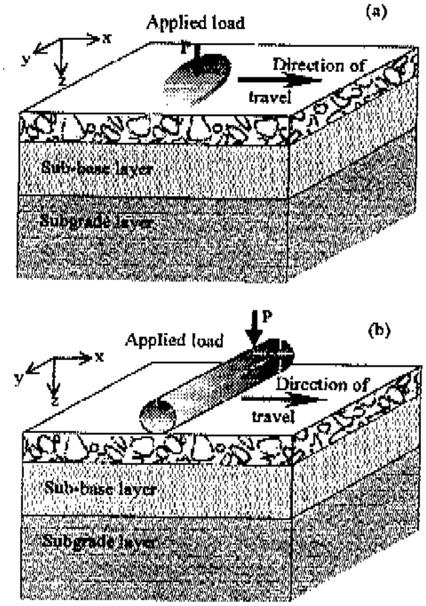

FIG. 2. Circular Patch and Strip Loadings

$$\sigma_{ij} = \lambda\sigma^e_{ij} + \sigma^R_{ij} + \rho_{ij} \qquad (2)$$

does not violate the yield condition anywhere in the pavement. Hence, by the basic maximum work inequality for perfectly plastic materials (Lubliner 1990), the virtual plastic work cannot be less than the virtual work done by the total stress field in (2) with the assumed kinematically admissible velocity field; that is

$$\sigma^*_{ij}e^*_{ij} = D(e^*_{ij}) \ge (\lambda\sigma^e_{ij} + \sigma^R_{ij} + \rho_{ij})e^*_{ij} \qquad (3)$$

where D = dissipation rate. If one integrates this inequality over any plane A $(x = x_0)$ and uses the principle of virtual work, the contributions to the integrals from the last two terms on the right-hand side are found to vanish, because σ^R_{ij} and ρ_{ij} have zero tractions on the pavement surface, so that

$$\int_A \sigma^*_{ij}e^*_{ij}\,dA \ge \lambda \int_A \sigma^e_{ij}e^*_{ij}\,dA \qquad (4)$$

It follows that if this inequality is violated, that is, if

$$\lambda > \frac{\int_A \sigma^*_{ij}e^*_{ij}\,dA}{\int_A \sigma^e_{ij}e^*_{ij}\,dA} \qquad (5)$$

then shakedown cannot occur. Values of λ obtained by applying the equality in (5) hence provide upper bounds to the shakedown load. The numerator in this expression is the internal plastic dissipation rate, as in conventional limit analysis calculations, and the denominator is the virtual "elastic dissipation rate" obtained by multiplying the elastic stresses with the plastic strain rates. The bound on λ is then given by the ratio of these two dissipation rates. It is important to appreciate that the value of the shakedown load is independent of the initial residual stress field σ^R_{ij}. The effect of this initial stress field is just to change the ultimate residual stress field ρ_{ij}. For a given mechanism the best bound is obtained by varying x_0 to find the smallest predicted value of λ.

The proof of this theorem assumes the material is perfectly plastic, satisfies Drucker's postulate, and thus has a "normal" flow rule and neglects moisture content. The consequences of relaxing these restrictions are discussed in later sections.

The above proof also neglects the weight of the pavement material. If this is included, the initial residual stress field must satisfy the equilibrium equation

$$\frac{\partial \sigma^R_{ij}}{\partial x_i} + \gamma_j = 0 \tag{6}$$

where γ_j = specific weight of the material (directed in the z-direction). On applying the virtual work identity to the integral of (3), an extra term appears reflecting the virtual work done by gravity, so that inequality (5) is replaced by

$$\lambda > \frac{\int_A \sigma^*_{ij} e^*_{ij}\, dA + \int_A \gamma_j v^*_j\, dA}{\int_A \sigma^e_{ij} e^*_{ij}\, dA} \tag{7}$$

SHAKEDOWN LOADS FOR COULOMB MATERIALS

General Procedure

The material in the various pavement layers is modeled as an elastic/perfectly plastic material failing according to Coulomb's criterion, characterized by a cohesion c and angle of internal friction ϕ. To facilitate the computation of these upper bounds, we adopt the approach widely used in limit analysis (Chen 1975) of considering failure mechanisms consisting of sliding or rotating rigid blocks. In such mechanisms, energy

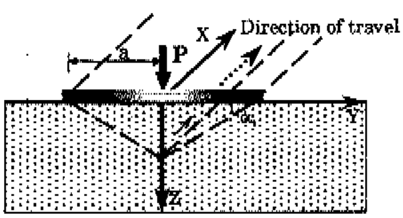

FIG. 3(a). Failure by Slipping along Channel

is only dissipated on the interfaces between the moving blocks and can be evaluated with relative ease. Because the loading history is independent of x—the coordinate in the travel direction—any assumed mechanism must also be independent of x. In the early studies these mechanisms consisted of sliding along channels under the wheel load in the travel direction as in Collins and Cliffe (1987) and Collins et al. (1993a,b) [Fig 3(a)]. Although such subsurface slip failures are observed in practice, rut formation is a much more common mode of failure. Hence recent research has concentrated on failure mechanisms that are plane strain in the (y, z)-plane and that involve the pavement sinking downward immediately under the load. Six such possible mechanisms have been investigated and are shown in Fig. 3(b). It is to be emphasized that the proof of the upper-bound shakedown theorem given above relates to these deformations that are independent of x and not to fully 3D deformation modes.

The plastic energy dissipation rate is

$$D^p = -\sigma_n[v_n] + |\sigma_t|[v_t] \tag{8}$$

per unit length of a discontinuity, where σ_n and σ_t are the normal and tangential stress components, respectively, with compressive stresses being taken as positive; and $[v_n]$ and $[v_t]$ denote the jumps in normal and tangential velocity, respectively. As a consequence of the normal flow rule assumption, the jump in the total velocity across such a discontinuity line must make an angle ϕ with this line, Chen (1975), so that

$$[v_n] = \tan\phi [v_t] \tag{9}$$

Because the stress components satisfy Coulomb's condition

$$|\sigma_t| = c + \sigma_n \tan\phi \tag{10}$$

(8) can be rewritten

$$D^p = c[v_t] \tag{11}$$

The numerator in the basic ratio (5) can hence be obtained by multiplying (11) by the length of each discontinuity and summing over all such discontinuities. The virtual "elastic" dissipation in the denominator in (5) can be calculated in the

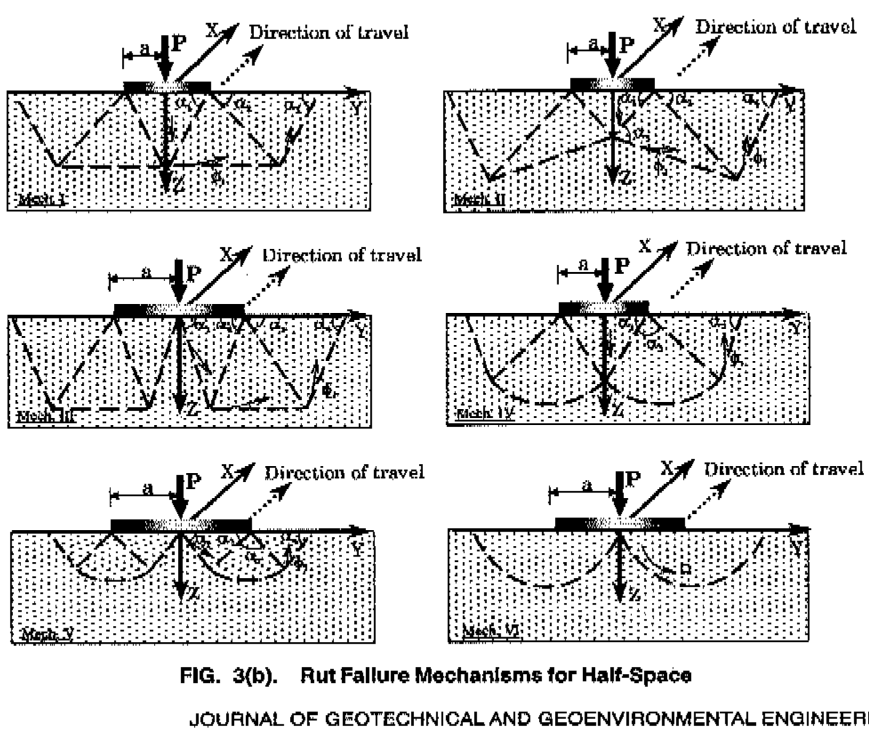

FIG. 3(b). Rut Failure Mechanisms for Half-Space

same way except that the total stresses in (10) are replaced by the elastic stresses produced by the applied load P.

Hence (11) is replaced by

$$D^e = c^e[v_i] \tag{12}$$

where

$$c^e = |\sigma_t^e| - \sigma_n^e \tan \phi \tag{13}$$

is termed the elastic cohesion by analogy with (10). In the calculations reported here the elastic stress field was calculated using the BISAR program, the applied pressure being assumed uniform over the loading patch. Surface friction in the travel x-direction is included but not in the lateral y-direction. The upper bounds to the shakedown value of the load parameter can be hence evaluated from

$$\lambda = \frac{c \sum_i [v_i]_i l_i}{\sum_i \int c_i^e [v_i]_i \, dl_i} \tag{14}$$

where l_i = length of the ith discontinuity. Each of the family of failure modes shown in Fig. 3 has one or more defining angles. These angles can be varied, together with the coordinate x_0 defining the position of the (y, z)-plane of deformation, and the ratio in (14) optimized to find the minimum value of λ associated with given family of mechanisms. These minima have been found using the simulated annealing procedure devised by Goffe et al. (1994). The standard Newton optimization procedure was found to be ineffective due to the large number of local minima.

Shakedown of Half-Spaces

The results of these calculations for a single uniformly loaded circular patch of radius a, with mean pressure p on a uniform half-space, are presented in Fig. 4, where the dimensionless load parameter

$$\mu_s = P_s/\pi a^2 c = p_s/c \tag{15}$$

is given for various values of the angles of internal friction. The variables $P_s = \lambda P$ and $p_s = \lambda p$ denote, respectively, the values of total load and mean pressure at which shakedown occurs. It is seen that there is not much to choose between the various modes for small friction angles <20°, but at higher angles the Prandtl type mechanisms IV and V give appreciably lower bounds. Mechanism V with a log-spiral fan failure zone gives the best results. Although this mechanism is similar to the well-known solution for failure of a foundation under plane strain conditions [e.g., Atkinson (1981)], the detailed shape of the optimal solution is very different. The deformation region does not spread out to the sides of the loading patch. Instead the triangles adjacent to the free surfaces are very narrow, and the upward motion is confined to a very narrow region adjacent to the loaded area as shown in Fig. 5. The shakedown theory hence predicts that the rut will have a narrow lip close to the edge of the wheel track. This is analogous to the formation of thin wear particles extruded sideways during some sliding processes on metal surfaces as observed by Kapoor (1997).

A comparison of the critical shakedown load factor predicted by the original channel solution with the above rutting solution is given in Fig. 6. These include the effect of tangential as well as normal surface loads. The values of μ are plotted against the surface coefficient of friction μ_l. It is seen that for values of $\mu_l < 0.2$ the rutting solution predicts a lower bed, but for higher tangential surface loads, failure by channel slipping is predicted.

It is to be emphasized that for a half-space (i.e., a single layer pavement model) the dimensionless load parameter μ depends on the angle of internal friction ϕ, the surface coefficient of friction μ_l, and Poisson's ratio ν. The value of this dimensionless shakedown load parameter does not depend on the value of the elastic modulus E, even though E/c could be viewed as a problem-defining dimensionless parameter. This is because the calculation of the shakedown load requires the computation of the elastic stress field [in the denominator of (5)] but not that of the elastic strain field. The elastic stress field can be deduced directly from the applied loading distribution and in axial symmetry involves ν but not E. (In the above calculation the value of ν was taken to be 0.3.) Thus the role of the elastic modulus is far less significant in the shakedown analysis than it is in the conventional model where failure is governed by an elastic strain rather than a stress. We note in passing however that if the value of E is too low, the

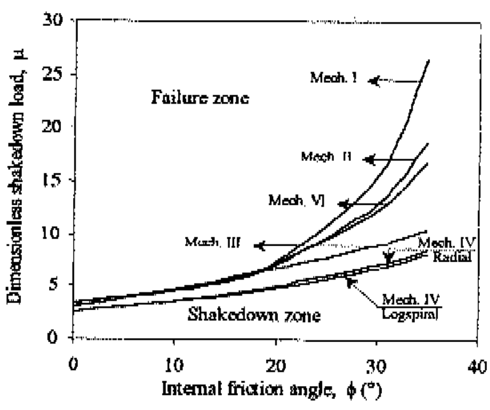

FIG. 4. Comparison between Different Mechanisms

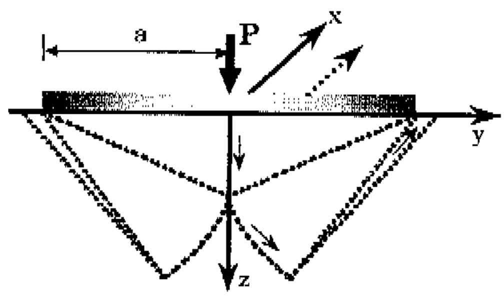

FIG. 5. Optimum Mode of Failure for $\phi = 0°$

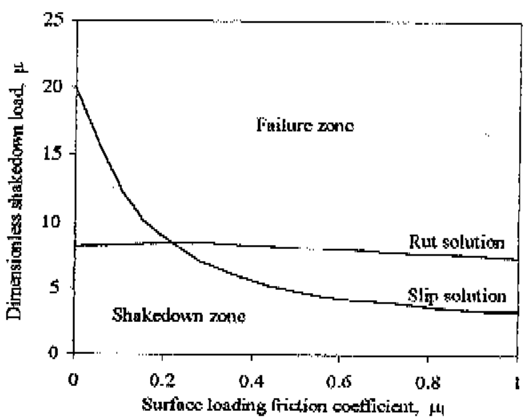

FIG. 6. Variation of Load Factor with Surface Friction for Slip and Rut Solutions

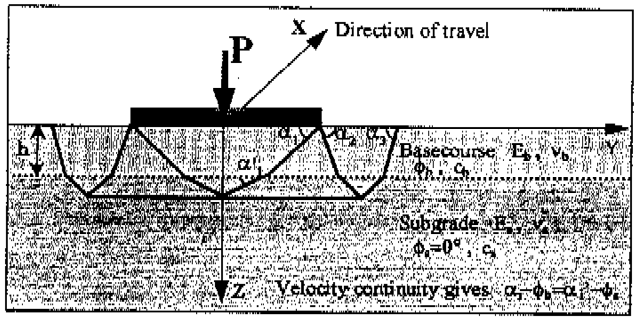

FIG. 7. Proposed Wedge Mechanism for Pavement Analysis

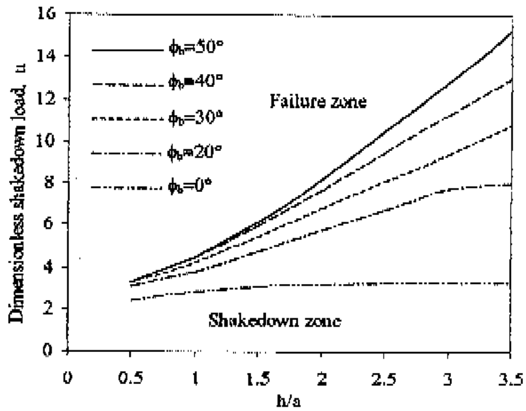

FIG. 8. Effect of h/a Ratio on Shakedown Load for $c_b/c_s = 1.75$ and for Various Basecourse Friction Angles

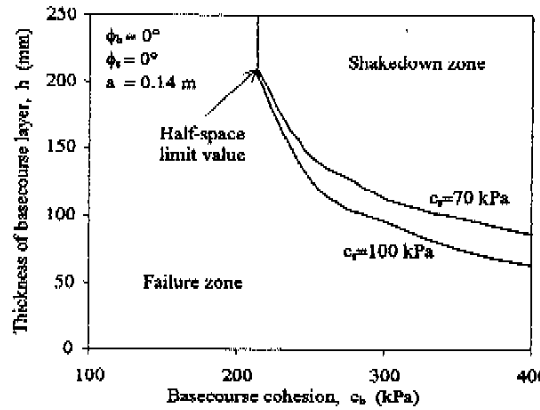

FIG. 10. Variation of Basecourse Thickness with Basecourse Cohesion

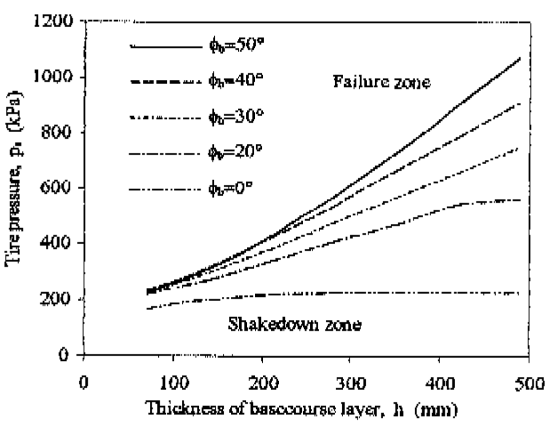

FIG. 9. Variation of Tire Pressure with h for Various Friction Angles ($c_b = 70$ kPa, $c_s = 40$ kPa, and $a = 0.14$ m)

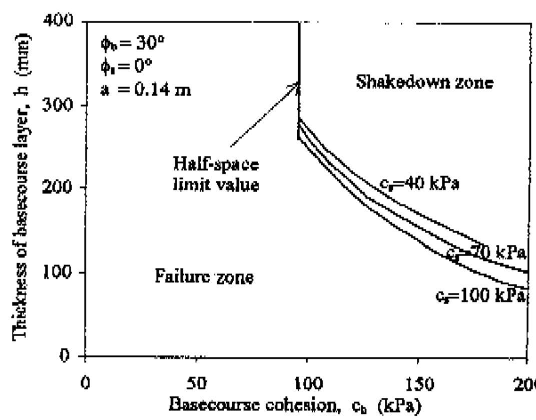

FIG. 11. Variation of Basecourse Thickness with Basecourse Cohesion

change in the shape of the contact region is more significant and cannot be ignored.

Shakedown of Layered Pavements

In practice, many rutting failures occur in pavements with weak subgrades. To model such failures the above procedure must be extended to layered pavements. An example of a possible mechanism for a pavement with a single basecourse of depth h is shown in Fig. 7. The discontinuity lines must have a change in slope as they cross the basecourse/subgrade boundary whenever the value of the internal friction angle is different in the two layers. This is also a feature of the limit analysis solution of Michalowski and Shi (1995) for the failure of footings on layered soils.

For the two-layer pavement model the dimensionless shakedown load parameter μ, defined to be $P_s/\pi \cdot c_b \cdot a^2$, depends on a number of dimensionless parameters, namely

$$\mu = \mu(\phi_b, \phi_s, \mu_t, \nu_b, \nu_s, E_b/E_s, c_b/c_s, h/a) \qquad (16)$$

In all the calculations presented here $\phi_s = 0$; thus the subgrade is assumed to be a saturated clay, and its cohesion c_s can be interpreted as its undrained shear strength, while $\mu_t = 0$, $\nu_b = 0.35$, $\nu_s = 0.4$, and $E_b/E_s = 3$. We hence concentrate on the effect of the major design parameters—the angle of internal friction of the basecourse material ϕ_b, the cohesions of the two layers c_b and c_s, and the thickness of the basecourse h.

The variation of μ with h/a for various basecourse friction angles for a fixed cohesion ratio is shown in Fig. 8. Graphs

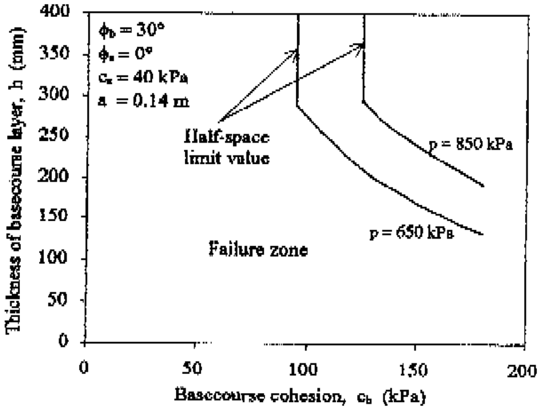

FIG. 12. Variation of Basecourse Thickness with Basecourse Cohesion for Different Tire Pressures

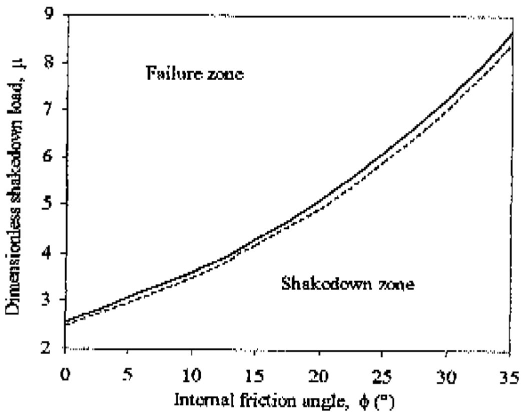

FIG. 13. Effect of Self-Weight of Soil on Shakedown Load Using Mechanism IV for $\gamma a/c = 0.031$

such as that shown in Fig. 8 enable the critical basecourse thickness at which shakedown just occurs to be determined for given values of tire pressure, radius of loaded area, and basecourse and subgrade strength parameters. Some representative dimensional results are given in Figs. 9–12. Fig. 9 is a dimensional form of Fig. 8 for specific cohesions and size of loaded area. Figs. 10–11 enable the basecourse thickness to be deduced for given values of the cohesions.

In Fig. 10 c_b, as well as c_s, is zero corresponding to undrained failure in the subgrade. Figs. 10 and 11 exhibit "cutoffs" at points where the cohesion of the basecourse is sufficiently low for failure to occur in the basecourse alone as predicted by the half-space solutions of the previous section. Although most design procedures are based upon tire pressures of 550–650 kPa, there are many instances when significantly higher pressures, up to 950 kPa, occur [e.g., Chowdhury and Rallings (1994)]. The effect of increasing this pressure on the predicted thickness is illustrated in Fig. 12.

EXTENSION OF BASIC MODEL

Effect of Self-Weight

If the self-weight of the aggregate is included, the calculation must be modified by the addition of the extra term in (7) reflecting the rate of work of gravitational forces. The extra term in the numerator of (7) is calculated by summing over each triangle as in standard limit analysis [e.g., Michalowski and Shi (1995)]. For a single layer pavement the presence of self-weight introduces another dimensionless problem parameter $\gamma a/c$, where γ is the unit weight of soil. The effect of self-weight on the shakedown load is very small as shown in Fig. 13, where results are plotted for $\gamma = 20$ kN/m^3, $a = 0.14$, and $c = 90$ kPa.

Effect of Dual Loads

In practice, loads are frequently applied through dual wheels, and the question arises as to whether the value of the shakedown load on one wheel is significantly affected by the presence of the adjacent load. Two possible types of failure mechanism are shown in Fig. 14. In the first the two wheels act separately, whereas in the second the two wheels effectively act as a single loaded area. Since in the optimal solutions the deforming region is confined to the edge of the loaded area, it is found that in fact there is very little interference between the deformation modes, and the mechanism in Fig. 14(a) is always the more critical. Nevertheless, the presence of a neighboring loaded patch does affect the value of the shakedown load for a given tire, because the elastic stress field

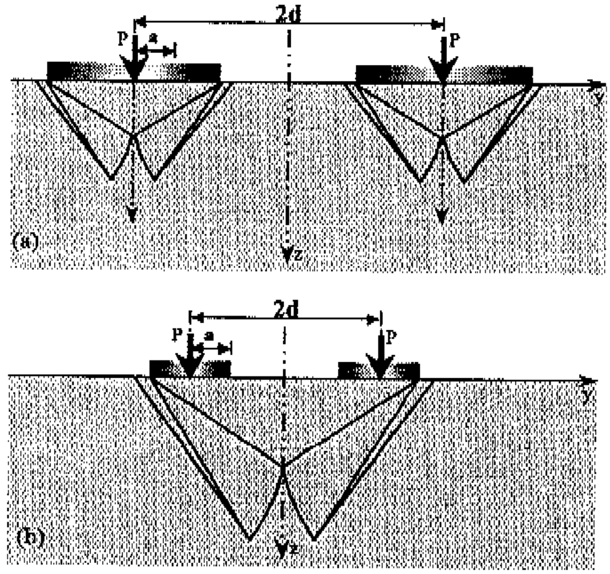

FIG. 14. Two Possible Mechanisms for Failure under Dual Loads

beneath the tire is now modified. This effect is illustrated in Fig. 15(a), where the shakedown loads for a single tire with load P is compared with that for a dual system with a load P on each tire. The presence of the neighboring load is seen to increase the shakedown load slightly—an effect more noticeable at higher friction angles.

It is also of interest to compare the shakedown load corresponding to a load P on a single tire with that of $P/2$ on each wheel of a dual wheel system. This is illustrated in Fig. 15(b), where $\tilde{\mu} = \lambda P_t/\pi \cdot c \cdot a^2$, with P_t being the total load on the system, is now plotted against ϕ. [Note that $\tilde{\mu} = \mu$ for a single tire, but $\tilde{\mu} = 2\mu$ for the dual wheel system.] It is seen that, as expected, the value of the shakedown load is significantly increased (more than doubled in fact) by splitting the load between the two tires.

Nonassociated Behavior

The fundamental shakedown theorems, on which these calculations are based, presume that the plastic flow rule is "associated" or "normal" (i.e., the plastic strain-rate vector is directed along the outward normal to the yield surface). In a Coulomb model material this means that the dilation angle ψ is equal to the angle of internal friction ϕ. As is well known

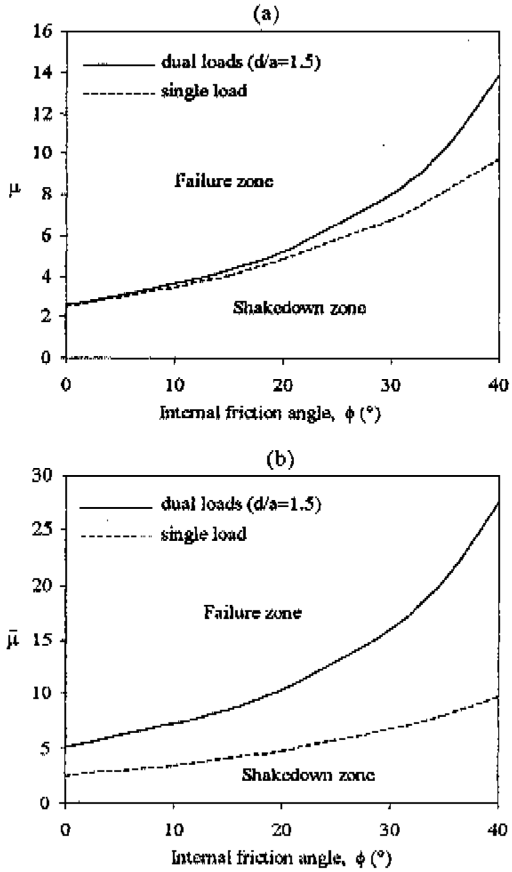

FIG. 15. Comparison of Values of Dimensionless Load Factors for Single and Dual Loads

this is not true for granular and aggregate materials where the value of ψ is appreciably less than ϕ.

For nonassociated materials the shakedown load, like the limit load is no longer unique. However it is still true that an upper bound obtained by assuming $\psi = \phi$ is still an upper bound to the shakedown load(s) for a material with $\psi < \phi$. This result, which parallels the well-known result for limit analysis, has been proved by Maier (1969) and Pycko and Maier (1995). The results presented above are hence still valid upper bounds, through they are not likely to be as good as bounds for an associated material.

Any attempt to directly apply a work-calculation, as described in the previous section, to a nonassociated material is thwarted by the fact that one needs to know the actual stress distribution in the pavement. However, Drescher and Detournay (1993) have shown that, for plane-strain deformation modes of the block sliding type, it is still possible to obtain load estimates by introducing a "comparison associated material" with cohesion and internal friction angle given by

$$\bar{c} = \omega c; \quad \tan \bar{\phi} = \omega \tan \phi \qquad (17a,b)$$

where

$$\omega = \cos \phi \cos \psi/(1 - \sin \phi \sin \psi) \qquad (18)$$

(Note that for an incompressible material with $\psi = 0$, $\omega = \cos \phi$ and $\tan \bar{\phi} = \sin \phi$.)

This procedure has been used here to calculate the reduced value of μ for nonassociated materials as shown in Fig. 16 for single layer pavements with ϕ in the range of 30° to 50°, and ψ in the range of 0 to ϕ.

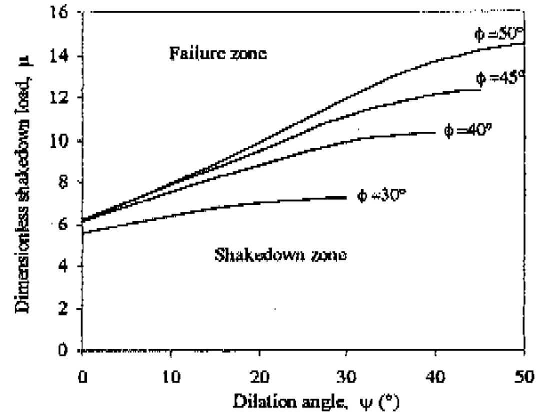

FIG. 16. Variation of μ against ψ for Homogeneous Semi-Infinite Pavement

EFFECT OF MOISTURE CONTENT

Pavements frequently fail as a result of water infiltrating the basecourse aggregate and subgrade. It is hence essential to be able to include the effect of moisture content in any pavement failure model, and the total stress must now be regarded as the sum of the pore-water pressure $\mu\delta_{ij}$ and the effective stress σ'_{ij}.

Drained Behavior

If the pavement is assumed to be fully saturated, the initial residual stress field σ_{ij}^R in the basecourse can be regarded as the sum of an effective residual stress $\sigma_{ij}^{R'}$ carried by the aggregate and the pore pressure $u\delta_{ij}$ by the water. We will assume that these two components equilibrate the weight of the aggregate and water, respectively, so that

$$\frac{\partial \sigma_{ij}^{R'}}{\partial x_i} + \gamma_j^s = 0; \quad \frac{\partial u}{\partial x_j} + \gamma_j^w = 0 \qquad (19a,b)$$

where γ_j^s and γ_j^w = self-weight of the solid and water phases, respectively.

Because the yield condition must now be formulated in effective stress space, the maximum work inequality [(3)] must be rewritten

$$\sigma_{ij}^{*\prime}\dot{\epsilon}_{ij}^{*} = D(\dot{\epsilon}_{ij}^{*}) \geq (\lambda\sigma_{ij}^{e} + \sigma_{ij}^{R'} + p_{ij})\dot{\epsilon}_{ij}^{*} \qquad (20)$$

The pore pressure, assumed constant in a drained deformation, does not enter this inequality. Upon integration, the basic inequality [(7)] is obtained again [or (5) if the self-weight of the aggregate is neglected]. The previous calculations can hence also be interpreted as those for drained loadings of saturated aggregates.

Undrained Behavior

In an undrained deformation of a saturated soil, the volume is preserved, and the pore pressure does no work. In the present context, such conditions are modeled by assuming a nonassociated material with zero dilation angle as described in the previous section. The value of the dimensionless shakedown parameter for a half-space is significantly reduced, particularly at large friction angles as shown in Fig. 17.

Unsaturated Behavior

In reality, the various layers of a pavement are neither fully saturated nor perfectly dry. The importance of the matric suction in determining the effective strength of partially saturated subgrades and basecourses is well understood and stressed by

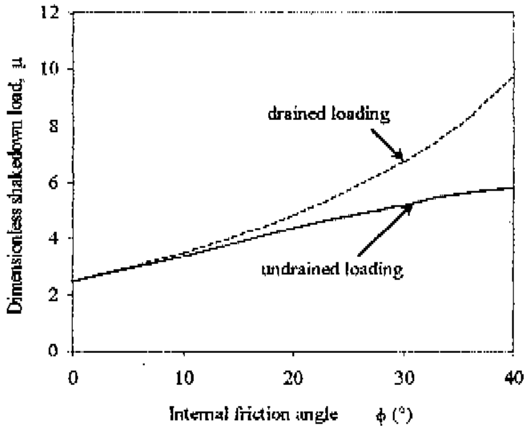

FIG. 17. Comparison of Shakedown Loads for Drained and Undrained Loadings

many authors [cf. Brown (1996) and Oloo et al. (1997)]. The effect of this suction is to pull particles together and significantly increase the effective cohesion of the aggregate or soil.

In the context of a Mohr-Coulomb model, the shear strength of an unsaturated soil can be expressed by generalizing (10) as follows (Fredlund et al. 1978):

$$|\sigma_t| = c + (\sigma_n - u_a)\tan\phi + (u_a - u_w)\tan\phi^b \qquad (21)$$

where u_a and u_w = air and water pressures, respectively; $\sigma_n - u_a$ = net normal stress; $u_a - u_w$ = matric suction; and ϕ^b = second friction angle associated with the matric suction. The "total cohesion" that describes the shear strength of the partially saturated layer is hence given by

$$\bar{c} = c + (u_a - u_w)\tan\phi^b \qquad (22)$$

The previous calculations of the dimensionless shakedown load μ can hence now be used to design pavements operating under known partially saturated conditions simply by replacing c by $\bar{c}$ in the calculation of the dimensional variables.

SUMMARY AND CONCLUSIONS

It has been demonstrated that the concepts of shakedown theory can, when applied via the upper bound theorem, provide a rational approach to the analysis of unbound pavements which

- Includes the plastic as well the elastic behavior of the various layers in the pavement.
- Incorporates the cyclic (repeated) nature of the loading of the pavement structure.
- Allows different forms of pavement failure, such as rut formation and subsurface slip to be studied in detail.
- Leads to a design procedure in which the basecourse thickness can be deduced as a function of the applied load and the strength and stiffness properties of the subgrade and basecourse.
- Enables the effect of a number of parameters, such as moisture content, spacing of dual loads, and self-weight to be systematically studied. Other possibilities not included here are the nature of the applied pressure distribution, lateral and edge constraints, the use of geotextiles, and the effectiveness of stabilization layers.

The main limitation of the model is the use of the nonhardening, perfectly-plastic, Mohr-Coulomb model. There are two avenues of research, which would make the procedure more realistic:

- Continue with this perfectly-plastic model but regard it as describing the ultimate failure state of a hardening/softening material as in the classical critical state theories or in the general family of models for geomaterials developed by Lade (1984). Bonaquist and Witczak (1997) have recently developed such a model, based on the Mroz et al. bounding surface model (1978) for the response of a soil subjected to repeated loadings. These researchers end up with Mohr-Coulomb and Drucker-Prager descriptions of the ultimate failure surface, which hence provide a basis for calculating the effective strength parameters needed in the shakedown analysis. Because shakedown theory describes the ultimate behavior of a structure, it is pertinent to use constitutive models that describe the ultimate behavior of the pavement layers.
- Extend the classical shakedown theory to the more complex plasticity models that describe the hysteretic behavior of soils in response to cyclic loads, of which a few are available in the literature.

Despite the shortcomings, it is argued that the concepts and techniques of shakedown theory has much to offer the pavement engineers. It is the only procedure currently available that incorporates both the nonuniform, elastic-plastic stress and deformation fields induced beneath the moving loads and recognizes that these fields will change with each load application until either ultimate failure occurs or the pavement shakes down.

ACKNOWLEDGMENTS

The writers are grateful to F. Bartley of Bartley Consultants New Zealand and R. Rallings of Pitt and Sherry Consultants Australia for helpful discussions and to the New Zealand Foundation for Research in Science and Technology and Transit New Zealand for financial support.

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APPENDIX II. NOTATION

The following symbols are used in this paper:

- a = radius of circular loaded area;
- c = cohesion of half-space material;
- c_b, c_s = basecourse and subgrade cohesions, respectively;
- c^e = elastic cohesion;
- D^e, D^p = elastic and plastic energy dissipation rates, respectively;
- d = half distance between centers of two circular loaded areas;
- E = Young's modulus;
- e^*_{ij} = virtual strain rate field;
- h = thickness of basecourse;
- l_i = length of ith discontinuity;
- P, q = applied load and tire pressure, respectively;
- P_s, q_s = applied load and tire pressure at shakedown limit, respectively;
- u_a, u_w = pore air and water pressures, respectively;
- v^*_i, $[v_n]$, $[v_t]$ = virtual velocity field and jumps in normal and tangential velocity components, respectively;
- γ_s = unit weight of soil;
- γ_w = unit weight of water;
- δ_{ij} = Kronecker delta symbol;
- λ = scalar load parameter;
- μ = dimensionless shakedown load parameter;
- μ_t = surface friction coefficient;
- ν = Poisson's ratio;
- ν_b, ν_s = Poisson's ratios for basecourse and subgrade, respectively;
- ρ_{ij} = induced residual stress field;
- σ^R_{ij} = initial residual stress field;
- $\sigma^{R'}_{ij}$ = residual effective stress field;
- σ^*_{ij} = virtual plastic stress field;
- σ^e_{ij}, σ_{ij}, σ'_{ij} = elastic, total, and effective stress fields, respectively;
- σ_n, σ_t = normal and tangential stress components, respectively;
- ϕ, ψ = internal friction and dilation angles, respectively;
- ϕ^b = friction angle associated with matric suction; and
- ϕ_b, ϕ_s = internal friction angles of basecourse and subgrade, respectively.